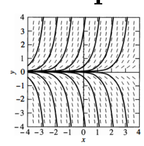
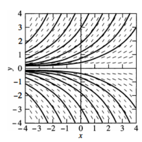
1.3: Slope Fields and Solution Curves

The aim of this section is to give a simple geometric way to think about solutions of a given (first-order) differential equation $y' = f(x,y)$. At each point (x,y) in the Cartesian plane ($\mathbb{R}^2$), the value of $f(x,y)$ determines a slope $m := y' = f(x,y)$. A solution of the differential equation is simply a differentiable function $y = y(x)$ that has the correct slope at each point $(x, y(x))$. Because of this we sometimes refer to the solution of the differential equation as the ________________. Furthermore, the graphical method we are going to use in this section (putting a line segment of slope m at each integer point (x,y)) creates what is called a **slope field** (or a **direction field**) for the equation $y' = f(x,y)$.

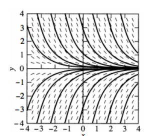
Example 1.



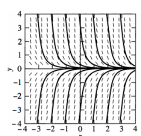
Slope field and solution curves for y'=2y.



Slope field and solution curves for y'=(0.5) y.



Slope field and solution curves for y'=-y.



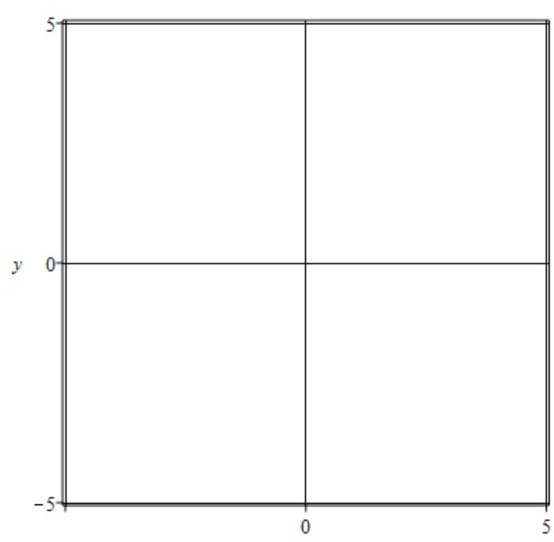
Slope field and solution curves for y'=-3y.

To the left is the slope fields and solutions curves for the differential equation

$$\frac{dy}{dx} = ky,$$

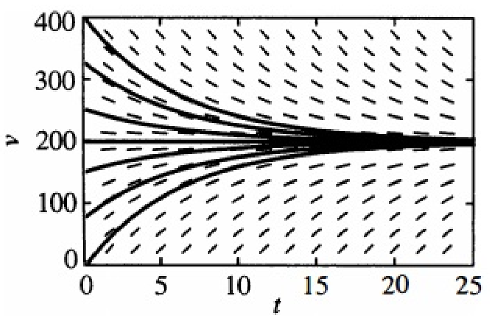
for $k = 2$, 0.5, -1, and -3. Spend a few minutes trying to understand these four pictures and then move on to Exercise 1 and give it a try yourself.

Exercise 1. Construct a slope field for the differential equation $y' = x - y$ and use it to sketch an approximate solution curve that passes through the point $(-4, 4)$.



You can already see from this one exercise that constructing slope fields is a tedious task by hand. However, many computer algebra systems have commands which will compute them for you. So let us not dwell on this and let us rather move on to some applications.

Example 2.

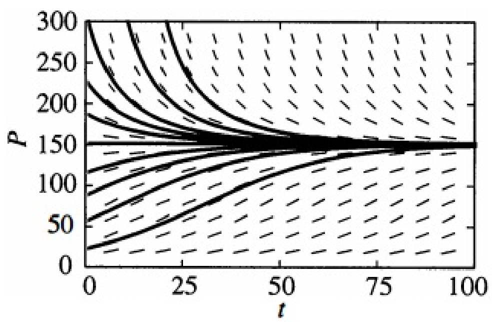


Suppose you throw a baseball straight downward from a helicopter hovering 3000 feet above ground. You wonder whether someone standing on the ground could possible catch the ball. Then the differential equation is given by

$$\frac{dv}{dt} = 32 - 0.16v_0 \quad \text{ft/s}.$$

You will notice that the limiting velocity is $v = 200$ ft/s $\equiv 136.36$ mi/h.

Example 3.



Later we will consider in detail the logistic differential equation

$$\frac{dP}{dt} = kP(M - P) \tag{1}$$

that is often used to model a population $P(t)$ with carrying capacity M.

Suppose that $k = 0.0004$ and $M = 150$ in (1). Then the slope fields and solutions curves are shown for different initial populations and we can see that the limiting population is $P = 150$.

Theorem 1. (Existence and Uniqueness of Solutions)
Suppose that both $f(x, y)$ and $D_y f(x, y)$ are continuous on some rectangle R containing (a, b). Then, for some interval I containing the point a, the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(a) = b$$

has exactly one solution on the interval I.

Homework. 1-5, 11-17 (odd)